Institution:
University of Lincoln

Unit of Assessment:
10 – Mathematical Sciences

Title of case study:
Building on Boole: Inspiring and Informing Public Understanding of Mathematical Research in Lincolnshire and Beyond

Period when the underpinning research was undertaken:
2016 - 2019

Details of staff conducting the underpinning research from the submitting unit:

Name(s):	Role(s) (e.g. job title):	Period(s) employed by submitting HEI:
KHUKHRO Evgeny	Professor of Pure Mathematics	1 Sep 14 to date
MATTAREI Sandro	Senior Lecturer/Programme Leader	1 Sep 15 to date
SMITH Simon	Senior Lecturer	1 Sep 16 to date
THILLAISUNDARAM Anitha	Lecturer	1 Sep 16 to date

Period when the claimed impact occurred:
2015 - 2019

Is this case study continued from a case study submitted in 2014?
N

1. Summary of the impact (indicative maximum 100 words)

Building on a rich local heritage of mathematics, University of Lincoln researchers have informed and shaped public attitudes to research in pure mathematics, through a series of lectures and school outreach events. This has fostered interest and enhanced engagement in mathematics research among school children (via Headstart Programme) and in rural communities who have limited opportunities to engage with research of this kind. These activities challenged the pervasive idea that all mathematics is known by highlighting the contribution of recent research in algebra conducted at the University of Lincoln, and featured in events for STEM at the Houses of Parliament, and talks to support strategies to encourage girls to study mathematics at university.

2. Underpinning research (indicative maximum 500 words)

Algebra is one of the foundations of mathematics and provides a language for applications of mathematics in physics, chemistry and other sciences. Research undertaken within the Charlotte Scott Research Centre for Algebra at the University of Lincoln is focused on group theory, which is the study of symmetries in abstract structures. Specifically, the research conducted includes the study of permutation groups, pro-p groups, number-theoretic aspects in groups and commutativity properties in groups; areas that have potential applications in cryptography and computer science.

The commutativity property in group theory is a wonderful topic for outreach, since it is so simple for school children to grasp: multiplication of numbers is commutative, but multiplication of matrices is not. Commutativity is one of the important properties a group can possess, and the research on this topic aims to achieve greater commutativity of important classes of groups and develop novel Lie rings methods for the study of groups. Among recent new results obtained by Khukhro are theorems on commutativity properties of groups and Lie rings admitting almost fixed-point-free automorphisms **[3.3].** This is a classical area of group theory and the theory of Lie rings, which still contains important open problems about automorphisms of composite order. Another set of results is about finite and profinite groups satisfying generalised Engel conditions, and novel applications of the famous Zelmanov theorem reductions **[3.1]**. A third result involves studying an important class of infinite groups in which restrictions have been placed on certain subgroups that control aspects of commutativity **[3.2]**.

Permutation group theory can be illustrated to non-specialists using diagrams of group actions. There is a natural way of decomposing permutation groups into smaller groups. Groups that are indecomposable in this way are called primitive and play a fundamental role in group theory. When taking a "product" of two groups, it seems intuitively that one should not end up with a primitive group. For approximately 100 years the only product known to preserve primitivity was the wreath product, but in **[3.5]** Smith discovered a new product, called the box product, that also preserves primitivity. The beauty of the box product, from the perspective of outreach, is that it can easily be described using attractive fractal-like diagrams.

Thillaisundaram's research concerns pro-p groups, and investigates their Hausdorff dimension, which is a generalisation of our usual concept of integer dimension to non-integer dimension. Hausdorff dimension was first applied in the 1930s to fractals and shapes in nature, and is a frequently used topic for outreach. For a given pro-p group, the Hausdorff dimension of a subgroup tells us how dense the subgroup is in the whole group. The collection of the Hausdorff dimensions of all subgroups in a given group is called a Hausdorff spectrum. The research **[3.6]** involves constructing pro-p groups with extremely infinite Hausdorff spectra, which has answered several open questions in the field, and motivated the development of further research areas.

Other research in the Charlotte Scott Research Centre for Algebra is of a number-theoretic nature, and studies power series whose coefficients are an integer sequence of some combinatorial relevance: this is often called the "generating function" of the sequence. Power series are an accessible topic for A-Level students. The particular research on which some outreach was based concerns generating functions that admit a closed form C(x) (which is, roughly, a "simple expression" for the sum of the series). Instances are known, or being discovered, where the finite sum obtained by discarding all terms of degree q or higher, with q a power of a prime p, also admits a closed form representation when viewed modulo p. Such a representation for the truncated sum modulo p frequently bears some resemblance with the shape of C(x), which however is only partial and often far from obvious. In **[3.4]** Mattarei and Tauraso have developed a method to directly infer the closed form representation of the truncated sum from the closed form of the series, for a class of series with coefficients of combinatorial significance.

3. References to the research (indicative maximum of six references)

3.1 E. I. Khukhro and P. Shumyatsky, Almost Engel finite and profinite groups, Int. J. Algebra Comput. 26, no. 5 (2016), 973–983.
http://eprints.lincoln.ac.uk/23232/1/23232%20khu-shu153.pdf
http://doi.org/10.1142/S0218196716500405

3.2 E. I. Khukhro, N. Yu. Makarenko, P. Shumyatsky, Locally finite groups containing a 2-element with Chernikov centralizer, Monatshefte fuer Mathematik 179, no. 1 (2016), 91–97.
http://eprints.lincoln.ac.uk/16203/7/16203%20khu-mak-shu-145.pdf
https://doi.org/10.1007/s00605-014-0701-8

3.3 E. I. Khukhro, N. Yu. Makarenko, P. Shumyatsky, Finite groups and Lie rings with an almost fixed-point-free automorphism of order 2^n, Proc. Edinburgh Math. Soc. 60 (2017), 391–412.
http://eprints.lincoln.ac.uk/17243/1/17243%20khu-mak-shu-144.pdf
https://doi.org/10.1017/s0013091516000225

3.4 S. Mattarei and R. Tauraso, From generating series to polynomial congruences, J. Number Theory 182 (2018), 179–205.
http://eprints.lincoln.ac.uk/28782/1/Truncation.pdf
http://dx.doi.org/10.1016/j.jnt.2017.06.007

3.5 S. M. Smith, A product for permutation groups and topological groups, Duke Math. J. 166, no. 15 (2017) 2965–2999.
http://eprints.lincoln.ac.uk/27788/1/A-new-product-Duke-UoL-Repository-Version.pdf
https://doi.org/10.1215/00127094-2017-0022

3.6 A. Thillaisundaram and B. Klopsch, A pro-p group with infinite normal Hausdorff spectra, Pacific J. Math. 303, no. 2 (2019), 569–603.
http://eprints.lincoln.ac.uk/36794/1/Normal_spectrum_PJM_181204.pdf
https://doi.org/10.2140/pjm.2019.303.569

4. Details of the impact (indicative maximum 750 words)

The School of Mathematics and Physics engages in a range of outreach activities to communicate the school's research and promote public interest in maths. Researchers in the Charlotte Scott Research Centre for Algebra have delivered a range of talks, activities and posters, aimed at members of the public, university students, school children and politicians. These have presented recent research breakthroughs from the centre in a way that was easily accessible to non-specialists. The overarching aim of these was to: (a) dispel the pervasive idea that all mathematics is known, (b) show that original research in pure mathematics remains important, and (c) show that important research in pure mathematics is taking place in Lincolnshire. The events below transformed the views of local, national and international audiences.

Engaging students and dispelling myths in the Headstart Programme. Headstart **[5.1]** is a programme run by the Engineering Development Trust (EDT), and is part of EDT's national strategy to encourage school pupils to have an interest and involvement in science and engineering. Headstart is for students aged 16-17. The School of Mathematics and Physics has sustained an ongoing relationship with the Headstart programme, and has hosted 30 Headstart students per year since 2017. Hosting it in Lincoln makes Headstart accessible for students coming from rural communities in the East Midlands. As part of this programme, Khukhro, Mattarei and Smith deliver lectures that cover their research breakthroughs in pure mathematics, using audience-appropriate methods to engage students. As noted in **[5.1]**:

i. Examples were used to explain Khukhro's commutativity-type conditions, to dispel the misconception that everything in pure mathematics is known and demonstrate to the audience the essence of a cutting-edge result in pure mathematics **[3.3]**.

ii. Smith's discovery of a new product for permutation groups ('box product') **[3.5]** was described in pictorial form to help students understand this complex phenomenon. The lecture focused on how dramatic research breakthroughs can happen simply by shifting one's perspective.

iii. The recursive structure of Pascal's triangle when viewed modulo a prime number was presented. Progressing from this, simple polynomial tools were developed in the lecture, which then allowed the audience to appreciate one key idea behind recent research by Mattarei and Tauraso **[3.4]**, relating certain types of power series with their truncated versions modulo a prime.

After attending the lectures, students' perception of mathematical research had changed **[5.1].** In particular, 61% of students became more interested in pure mathematics as a subject, 46% of students felt that being able one day to prove their own new theorem seemed more achievable, 79% of students felt that becoming a professional mathematician would be more achievable, and 93% of students felt that they had achieved a better understanding of what pure mathematics is.

Public Lectures.
The School of Mathematics and Physics organises several annual public lectures, named after notable figures in the history of mathematics and physics associated with Lincolnshire, including

George Boole, who was born in Lincoln and was one of the most influential mathematical minds of the 19th century, and Charlotte Scott, a 19th century algebraist who was influential in the development of American mathematics, including the mathematical education of women. These talks allow rural communities in Lincolnshire to engage with points (a)-(c) above. Khukhro has given two of these public lectures; audiences were members of the public and included many school children from Lincolnshire, with feedback **[5.2]** for events demonstrating clear evidence of public engagement with points (a)-(c).

- 2015, "George Boole: legacy of a mathematics revolutionary", (107 attendees). The lecture covered logic from its origins to modernity, and its applications to group theory. Central to this talk was Khukhro's **[3.2]** research that resulted from applying logic to group theory. An attendee wrote, "*The lecture showed the power of the mathematical advances that are made by researchers such as those at the Lincoln School. Pure mathematics can seem very esoteric to the layman, but the lecture showed…how it lays the foundation for all of our modern world.*" **[5.2].**

- 2018, "Is algebra a spoilsport in mathematics?", (119 attendees). Central to this talk was Khukhro's research **[3.1]** on commutativity-type conditions. Attendees wrote, "*opening up such a crucial subject to a broad audience in such a wonderful way is massively important for Lincoln*" **[5.2],** and "*I left with the knowledge that highly important algebraic innovations are in progress, at the University of Lincoln. I also understood, for the first time, that research in mathematics is still taking place, and that it is important*" **[5.2].**

Khukhro repeated the 2015 lecture in Brazil and Russia, bringing points (a)-(c) to an international audience.

- 2016, University Public Lecture, University of Brasilia, Brazil (40 attendees).

- 2017, University Public Lecture, University of Novosibirsk, Russia (40 attendees).

Houses of Parliament STEM for Britain Event.
In March 2019, Thillaisundaram presented her research **[3.6]** on Hausdorff dimensions of pro-p groups at the event STEM for Britain, organised by the Houses of Parliament Parliamentary and Scientific Committee, held in Westminster. Her poster included the origins of Hausdorff dimensions in the study of fractals and the leap made to applying Hausdorff dimensions to pro-p groups, highlighting points (a)-(c) above to the audience. The audience was mostly Parliamentarians (**[5.3]** and http://www.stemforbritain.org.uk/).

Women in Mathematics.
Many activities promoting women in mathematics have been organised by the Charlotte Scott Research Centre for Algebra. At these events, current research from the centre is used to highlight points (a)-(c) above. For example, in March 2017, Thillaisundaram organised a Newton Academy mathematics event for schoolgirls, held at the University of Lincoln. The event included talks on Thillaisundaram's research, with a problem-solving opportunity for the participants. In March 2019, Thillaisundaram spoke on her research on Hausdorff dimensions of pro-p groups (**[3.6]**) at the Bishop Grosseteste University Girls in Mathematics Day, which was sponsored by the London Mathematical Society. Thillaisundaram introduced fractals and the current open problems in the study of Hausdorff dimensions of pro-p groups. Several attendees of this event listed Thillaisundaram's session as their favourite (**[5.4]**).

5. Sources to corroborate the impact (indicative maximum of 10 references)

5.1 EDT Headstart testimonial letter.
5.2 Selection of Testimonials from Khukhro's Public Lectures.
5.3 Thillaisundaram's invitation letter to present a poster at STEM for Britain.
5.4 Testimonial from BGU Programme Leader for Mathematics for Girls in Mathematics Day.