Institution: University of Lincoln		
Unit of Assessment: 10 - Mathematics		
Title of case study: Connecting the Disconnected: Revising Curricula and Teaching Materials through New Insights into Infinite Groups		
Period when the underpinning research was undertaken: 2016 - 2017		
Details of staff conducting the underpinning research from the submitting unit:		
Name(s):	**Role(s) (e.g. job title):**	**Period(s) employed by submitting HEI:**
KHUKHRO Evgeny	Professor of Pure Mathematics	1 Sep 14 to date
SMITH Simon	Senior Lecturer	1 Sep 16 to date
Period when the claimed impact occurred: 2018 - 2021		
Is this case study continued from a case study submitted in 2014? N		

1. Summary of the impact (indicative maximum 100 words)

Locally compact groups are fundamental to mathematics and physics, and form a core part of many advanced undergraduate and MSc mathematics curricula. Until recently, these courses focused on locally compact groups that are "connected", relying on academic insights circa 1950, because very little was known about those which are "totally disconnected".

From discoveries in the 1990s a theory of totally disconnected locally compact ("tdlc") groups emerged, but by 2013 there remained a central problem: the theory depended heavily on understanding tdlc groups that are compactly generated and simple, but researchers did not know how many of these groups there were. In 2017, Smith solved this problem. By developing a new product for groups, Smith's research determined the number of these simple groups and showed that the preferred tool used by researchers at the time ("isomorphism") could not be used to categorise them. This caused a **paradigm shift** in the field, and **changes in undergraduate and postgraduate curricula** internationally.

Associated research by Khukhro gave new insights into the commutativity properties of groups and Lie rings and has been **incorporated into the syllabus of an advanced course on group theory** in Brazil.

2. Underpinning research (indicative maximum 500 words)

Algebra and group theory lie in the foundations of mathematics, providing structure to various mathematical theories, as well as giving the language for applications of mathematics in physics, chemistry and other sciences. The algebraic research in the University of Lincoln's Charlotte Scott Research Centre for Algebra is focused on group theory and the theory of Lie rings.

Within group theory, the study of (non-discrete) topological groups that are locally compact is a longstanding area of research. The study famously begins with a natural decomposition into the connected and totally disconnected cases. Pivotal research by Prof. George Willis in the 1990s opened up the field of totally disconnected locally compact groups, and whilst connected locally compact groups are broadly understood following the solution to Hilbert's 5th problem, totally disconnected locally compact groups are in comparison poorly understood, even those that are compactly generated. The key to understanding compactly generated tdlc groups lies in understanding those that are topologically simple, and so a central problem in the field was to determine (up to isomorphism) the number of such groups.

To address this, Smith applied his research on permutation groups to the study of compactly generated tdlc groups. In **[3.1]**, Smith solved this problem by developing a new product for permutation groups and topological groups, called the box product. The box product provides a general way to easily build exotic examples of (non-discrete) simple, tdlc, compactly generated topological groups from discrete groups. Using it, Smith showed that (up to isomorphism) there are precisely as many of these groups as there are real numbers, by constructing an uncountable family of simple, totally disconnected, compactly generated, locally compact groups. He also showed that classifying these simple groups up to isomorphism was impossible, leading to the field of tdlc research shifting its focus away from the isomorphism relation and towards something called a "local isomorphism".

Some of the most important properties in group theory are commutativity properties. Recent results obtained by Evgeny Khukhro and his coauthors are theorems on the commutativity properties of groups and Lie rings admitting almost fixed-point-free automorphisms **[3.2].** They for the first time obtained a bound for the derived length of a group with an automorphism of composite order in terms of its fixed points, which demonstrated for the first time a possibility for the application of Lie ring methods to a well-known problem about such bounds. Another set of results by Khukhro and his coauthor **[3.3]** introduced a new type of generalised Engel conditions for finite and profinite groups and developed novel methods for applications of Zelmanov's theorem and Hall–Higman type reductions.

3. References to the research (indicative maximum of six references)

3.1 Simon M. Smith, A product for permutation groups and topological groups, Duke Math. J. 166, No. 15 (2017) 2965–2999.
https://doi.org/10.1215/00127094-2017-0022
http://eprints.lincoln.ac.uk/27788/1/A-new-product-Duke-UoL-Repository-Version.pdf

3.2 E. I. Khukhro, N. Yu. Makarenko, P. Shumyatsky}, Finite groups and Lie rings with an automorphism of order 2^n, Proc. Edinburgh Math. Soc. 60 (2017), 391–412.
https://doi.org/10.1017/s0013091516000225
http://eprints.lincoln.ac.uk/17243/1/17243%20khu-mak-shu-144.pdf

3.3 E. I. Khukhro and P. Shumyatsky, Almost Engel finite and profinite groups, Int. J. Algebra Comput. 26, no. 5 (2016), 973–983.
http://doi.org/10.1142/S0218196716500405
http://eprints.lincoln.ac.uk/23232/1/23232%20khu-shu153.pdf

4. Details of the impact (indicative maximum 750 words)

Our research demonstrates a step change in academic understanding of group theory, with these conceptual changes enabling a number of changes within mathematics curricula at HEIs.

Impact on core understanding and theory

A paradigm shift in the core understanding of a subject necessitates changes in curricula. We begin by giving details of the impact of Smith's research on this core understanding. Smith's result led to the entire "*field shifting its focus away from global isomorphism relations…towards local isomorphism relations"* **[5.1].** It now forms part of the core theory of tdlc groups and features prominently in the main textbook on tdlc group theory **[5.2],** with an entire chapter devoted to it: the chapter is titled "Simon Smith's construction of an uncountable family of simple, totally disconnected, locally compact groups" (more detail below). Pierre-Emmanuel Caprace (one of the world's most highly-cited algebraists) gave a prestigious address to the 7th European Congress of Mathematics about the study of these simple groups, and wrote a survey paper for the proceedings. In the concluding remarks of the survey **[5.3, page 17]** he writes *"the study of [these simple groups] should not be aimed at a classification up to isomorphism: [Smith's result] provides strong evidence that this classification problem is [essentially impossible]*". As one of the co-authors of the textbook writes in his testimonial **[5.1],** Smith's result gives the "*correct*"

perspective on the theory of locally compact groups. This correction required HEIs to update their curricula; we give details below.

Impact on design and delivery of the mathematics curriculum in universities in the UK, Europe, USA, and Brazil.

New standard textbook for courses on totally disconnected locally compact groups has one chapter exclusively devoted to Smith's research [3.1]

The breakthroughs by Prof. George Willis in the 1990s in the theory of locally compact groups resulted in many universities running courses on totally disconnected locally compact (tdlc) groups. However, until 2018 there was no single textbook that could be used for such a course. This changed in 2018 with the London Mathematical Society (LMS) publication of a Lecture Note Series textbook entitled "New Directions in Locally Compact Groups", published by Cambridge University Press. This textbook was produced to drive forward the field of locally compact groups. In the textbook (which has 20 chapters), an entire chapter authored by Prof. George Willis (the pre-eminent expert on tdlc groups) and Dr Colin Reid is dedicated to Smith's work. Entitled "Simon Smith's construction of an uncountable family of simple, totally disconnected, locally compact groups" **[5.2],** it presents the results in Smith's paper **[3.1]**, which provide a fundamental change of perspective on these simple groups, causing the field to move "*its focus away from global isomorphism relations…towards local isomorphism relations*" **[5.1]**. This chapter forms a core part of a syllabus on tdlc groups because it gives the correct perspective, that "*the study of [these simple groups] should not be aimed at a classification*" using the standard tool in group theory, "*isomorphism*" [**5.3, page 17].**

Despite only being published in 2018, this textbook has already been assigned as recommended reading in a 2019 course for undergraduates and masters students at ETH Zürich **[5.4].** In **[5.1]**, one of the co-authors of the LMS textbook states that "*This textbook is now the textbook choice for a modern course on locally compact groups with a focus on tdlc groups. Any course that closely follows this textbook would include Smith's result prominently."*

Change in mathematical curricula and course requirements

In addition to being used in undergraduate mathematics curricula at the University of Lincoln (new Group Theory course MTH3003M focusing on permutation groups, **[5.5]**), the research has led to curricula and course changes in the UK and overseas.

a) Revised curricula at Binghamton University, USA, drawing on Smith's research **[3.1]**.
At Binghamton University, USA, a course entitled *Math 603A Locally Compact Groups* was delivered in autumn 2017. The curriculum of the course covered Smith's construction in **[3.1]** of an uncountable family of simple, compactly generated, totally disconnected, locally compact groups **[5.1].** As noted in **[5.1],** this result was presented to the students in order to give the "*correct*" perspective on the theory of locally compact groups.

b) Assessment question at the University of Oxford using Smith's research **[3.1]**.
At the University of Oxford, a course on Totally Disconnected Locally Compact Groups was given in Spring 2018 **[5.6]** (before publication of the LMS textbook). The course overview states that "*The goal of the course is to introduce the central ideas in this new theory [of tdlc groups]*". As evidenced previously, the box product construction in **[3.1]** by Smith is a central idea that gives the "*correct*" perspective on the theory. To obtain credit for the course at Oxford, students had to complete one of the four "Credit Options" **[5.7].** The 4th credit option is "*Define the box product first introduced in `A product for permutation groups and topological groups' by Simon Smith, and explain when the box product of two groups is a separable tdlc group.*"

c) Updated curricula on "Lie ring methods in group theory" at the University of Brasilia, drawing on Khukhro's work.
A course on Lie ring methods in group theory is an advanced course in group theory that can be found at many HEIs around the world. Typically, it is taught to postgraduate

students. Such a course ran at the University of Brasilia, Brazil, in the second semester 2018, taught by Dr Cristina Acciarri, attended by 5 PhD students and 3 MSc students. This course in Brasilia strived to bring the students up to the cutting-edge for results about Lie ring methods in group theory. Khukhro's work **[3.2, 3.3]** featured prominently in a revised curriculum of a course on "Lie ring methods in group theory" delivered at the University of Brasilia in 2018. Indeed, Dr Acciarri attests **[5.8]** "*when designing the syllabus we encountered two issues: …existing textbooks [only covered] well-established techniques [for tackling a well-known problem]*" and "*we needed…a source of new types of problems…appropriate for our students…[that] fit with the course material. The syllabus…was influenced by **[3.2]** and **[3.3]** because these papers helped us address these two issues."*

5. Sources to corroborate the impact (indicative maximum of 10 references)

5.1 Testimonial from a former Riley Visiting Assistant Professor at Binghamton University, USA and coauthor of the LMS Lecture Note Series textbook "New Directions in Locally Compact Groups".

5.2 Details of the LMS Lecture Notes Series textbook "New Directions in Locally Compact Groups", including table of contents (Chapter 7 is called "Simon Smith's construction of an uncountable family of simple, totally disconnected, locally compact groups").

5.3 Highlighted copy of paper: P-E. Caprace, `Non-discrete simple locally compact groups', published in the European congress of mathematics, Proceedings of the 7th ECM (7ECM) congress, Berlin, Germany, July 18–22, 2016, by the European Mathematical Society.

5.4 Pdf copy of course webpage for an ETH course on tdlc groups (original url: http://www.vvz.ethz.ch/Vorlesungsverzeichnis/dozent.view?dozide=10060888&semkez=2019S&ansicht=2&lang=en, accessed 18/3/21)

5.5 Course material from University of Lincoln module MTH3003M (Group Theory)

5.6 Overview of tdlc groups course given at University of Oxford in 2018 (see https://people.maths.bris.ac.uk/~cb20249/Teaching.html and https://people.maths.bris.ac.uk/~cb20249/TDLC_Overview.pdf, accessed 22/3/21)

5.7 Credit options for students sitting the tdlc groups course given at University of Oxford in 2018 (see https://people.maths.bris.ac.uk/~cb20249/Teaching.html and https://people.maths.bris.ac.uk/~cb20249/TDLC_credit.pdf, accessed 22/3/21)

5.8 Testimonial from University of Brasilia.